# Introduction to Topology

## Notations

| | |
|---|---|
| $2^X$ | **Power Set** of $X$ |
| $A - B$ or $A \backslash B$ | $\{x \mid x \in A, x \notin B\}$ |
| $A \amalg B$ | **Disjoint Union** of $A$ and $B$ |

## Basic Point-Set Topology

### Definition of Topological Space

set $X$ with a collection $\mathcal{O}$ of subsets of $X$, such that

- Both $\varnothing$ and $X$ are in $\mathcal{O}$
- The union of any number of sets in $\mathcal{O}$ is in $\mathcal{O}$
- The intersection of finite number of sets in $\mathcal{O}$ is in $\mathcal{O}$

Any $O \in \mathcal{O}$ is called open, and $X - O$ is closed.

For closed sets $A_k$, $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i \in S} A_i$ are closed

### Different Topologies of X

Trivial Topology $\mathcal{O} = \{\varnothing, X\}$

Finite Complement Topology $\mathcal{O} = \{O \mid O^c \text{ is finite}\} \cup \varnothing$

Euclidean(Usual) Topology

$X = \mathbb{R}^n, \quad \mathcal{O} = \{O \mid O \text{ is union of open } n\text{-balls}\} \cup \varnothing$

Discrete Topology $\mathcal{O} = 2^X$

### Interior, Closure and Boundary

Interior $\text{int}(A) = \{x \mid \exists \text{ open set } \mathcal{O}_x, \text{s.t.} x \in \mathcal{O}_x \subseteq A\}$

Exterior $\text{ext}(A) = \{x \mid \exists \text{ open set } \mathcal{O}_x, \text{s.t.} x \in \mathcal{O}_x \subseteq X - A\}$

Boundary $\partial A = \{x \mid \forall \mathcal{O}_x \ni x, \mathcal{O}_x \cap A \neq \varnothing, \mathcal{O}_x \cap (X–A) \neq \varnothing\}$

Closure $\overline{A} = \text{int}(A) \amalg \partial A$

$\text{ext}(A) = \text{int}(X - A), \text{int}(A) = \text{ext}(X - A)$

$X = \text{int}(A) \amalg \partial A \amalg \text{ext}(A)$

$\overline{A \cup B} = \overline{A} \cup \overline{B} \quad \text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$

### Basis for a topology

$\mathcal{B}$ where $\forall O \in \mathcal{O}, O = \bigcup_{B_k \in \mathcal{B}} B_k$

$$\left.\begin{array}{l} X = \bigcup_{B \in \mathcal{B}} B \\ \forall B_1, B_2 \in \mathcal{B} \text{ and } x \in B_1 \cap B_2, \\ \exists B_3 \in \mathcal{B}, \text{s.t.} x \in B_3 \subset B_1 \cap B_2 \end{array}\right\} \Rightarrow \mathcal{B} \text{ is a basis}$$

### Metric Spaces

Metric $d : X \times X \to \mathbb{R}$ such that

- $d(x, y) > 0$, and equality holds only if $x = x$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

p-norm for $x, y \in \mathbb{R}^n$, $d(x, y) = \left(\sum_{i=1}^{n} |a_i - b_i|^p\right)^{\frac{1}{p}} \ (p \geq 1)$

Metric Space topology on $X$ whose basis is a collection of all open balls $B_r(x) = \{b \mid d(b, x) < r\}$ with $r > 0, x \in X$

All p-norm are equivalent $\Rightarrow$ introduce the same topology

### Subspaces

$A \subset X$ then $\mathcal{O}_A = \{O \cap A \mid O \in \mathcal{O}_X\}$ is a subspace topology

For open set $A \subset X$, $B$ open in subspace $A \Leftrightarrow B$ open in $X$

For closed set $A \subset X$, $B$ closed in $A \Leftrightarrow B$ closed in $X$

### Continuity

$f : X \to Y$ is continuous if $\forall O \in \mathcal{O}_Y, f^{-1}(O) \in \mathcal{O}_X$

$f : X \to Y$ conti. $\Leftrightarrow f : X \to f(X)$ conti.

$f : X \to Y$ conti., $g : Y \to Z$ conti. $\Rightarrow gf : X \to Z$ conti.

$f : X \to Y$ conti., $A \subset X \Rightarrow f|_A : A \to Y$ conti.

### Homeomorphisms

A continuous bijective map $f : X \to Y$ with conti. inverse

### Product Spaces

Product Topology of $X \times Y$ has basis $\{U \times V \mid U \in \mathcal{O}_X, V \in \mathcal{O}_Y\}$

## Connectedness

### Connected

$X \neq$ union of two disjoint nonempty open sets

$\Leftrightarrow X \neq$ union of two disjoint nonempty closed sets

$\Leftrightarrow$ the only open and closed sets in $X$ are $\varnothing$ and $X$

An interval $[a, b]$ in $\mathbb{R}$ is connected

$A$ connected $\Rightarrow \overline{A}$ connected

$\{C_\alpha\}$ connected, mutually intersects $\Rightarrow \bigcup_\alpha C_\alpha$ connected

### Path-connected

$\forall a, b \in X, \exists \text{ conti.} f : [0, 1] \to X, \text{s.t.} f(0) = a, f(1) = b$

Path-connected $\Rightarrow$ connected

Suppose $f : X \to Y$ is conti. and onto, then

- $X$ connected $\Rightarrow Y$ connected
- $X$ path-connected $\Rightarrow Y$ path-connected

# Compactness

## Compact

$X$ is compact if every open cover of $X$ has a finite subcover

An interval $[a, b]$ in $\mathbb{R}$ is compact

$X$ is compact $\Rightarrow$ subset of $X$ is compact

$X$,$Y$ are compact $\Rightarrow$ $X \times Y$ is compact

$X \subset R$ is compact $\Leftrightarrow$ $X$ is closed and bounded

Suppose $f : X \to Y$ is conti. and onto, then

- $X$ compact $\Rightarrow$ $Y$ compact

## Hausdorff

$\forall$ different $x, y \in X, \exists$ disjoint open sets $U \ni x, V \ni y$

$X$ is hausdorff, $A \subset X$ is compact $\Rightarrow$ $A$ is closed

Properties of hausdorff spaces

- In a hausdorff space, points are closed sets
- A subspace of a hausdorff space is hausdorff
- Product space of two hausdorff spaces are hausdorff

## Normal

$\forall$ disjoint closed $A, B \subset X, \exists$ disjoint open $U \supset A, V \supset B$

$X$ is compact and hausdorff $\Rightarrow$ $X$ is normal